DENSITY IS AT MOST THE SPREAD OF THE SQUARE

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1.1 Claim. *Assume $\mathbb{B}$ is an infinite Boolean Algebra and $\lambda = d(\mathbb{B})$. Then $\mathfrak{s}(\mathbb{B} * \mathbb{B})$, i.e. $\mathfrak{s}(\mathrm{uf}(\mathbb{B}) \times \mathrm{uf}(\mathbb{B})) \geq \lambda$ (if λ limit-obtained).*

Remark. 1) $\mathrm{ul}(\mathbb{B})$ is the space of ultrafilters of $\mathbb{B}$ a compact space with clopen base.
2) $\mathfrak{s}(X)$ is $\sup\{|Y| : Y \subseteq X$ is discrete Y, same as $\mathrm{des}(X)\}$.
3) We meant to consider whether this works for compact Hausdorff spaces. But subsequently and independently Szentmiklössy prove this.

Proof. Without loss of generality $\lambda > \aleph_0$. We choose (p^0_i, p^1_i, a_i) by induction on $i < \lambda$ such that

$\otimes$ (a) p^ℓ_i is an ultrafilter of $\mathbb{B}$ for $\ell = 0, 1$
 (b) $a_i \in p^1_i, a_i \notin p^0_i$, i.e. $(-a_i) \in p^1_i$
 (c) if $j < i$ then $a_j \in p^0_i \Leftrightarrow a_j \in p^1_i$
 (d) if $j < i$ then $a_i \notin p^0_j, a_i \notin p^1_j$.

So assume we have arrived to i. Let $\mathbb{B}_i$ be the subalgebra of $\mathbb{B}$ generated by $\{a_j : j < i\}$.

For every non-zero $b \in \mathbb{B}_i$ choose an ultrafilter q^i_b of $\mathbb{B}$ and for simplicity $b = a_j \Rightarrow q^i_b = p^1_j$ and $b = (-a_j) \Rightarrow q^i_0 = p^0_j$ for $j < i$.

As $d(\mathbb{B}) \geq \lambda$ clearly $\{q^i_b : b \in \mathbb{B}_i \backslash \{0\}\}$ is not dense hence there is a non-zero $a_i \in \mathbb{B}$ such that $b \in \mathbb{B}_i \backslash \{0\} \Rightarrow a_i \notin q^i_b$ (i.e. a non-empty clopen set to which none of the points q^i_p belongs).

Now clearly $b \in \mathbb{B}_i \backslash \{0\} \Rightarrow a_i \neq b$ (as $b \in q^i_p$) hence $a_i \notin \mathbb{B}_i$. This implies that there is an ultrafilter q^*_i of $\mathbb{B}_i$ such that

$\circledast$ $b \in q \Rightarrow a_i \cap b > 0 \wedge (-a_i) \cap b > 0$.

[Why? As $\{b_0 \cup b_1 : b_0, b_1 \in \mathbb{B}_2$ and $b - 1 \cap a_i = 0_{\mathbb{B}}$ and $b_a - a_i = 0\}$ is a proper ideal of $\mathbb{B}_i$ hence can be extended to an ultrafilter of $\mathbb{B}_i$.]

So there are ultrafilters p^0_i, p^1_i of $\mathbb{B}$ such that

$\circledast$ $q^*_i \cup \{a_i\} \subseteq p^1_i$ and $q^*_i \cup \{-a_i\} \subseteq p^0_i$.

This is enough for the induction step.

Having carried the induction

(a) $p_i := (p^0_i, p^1_i) \in \mathrm{uf}(\mathbb{B}) \times \mathrm{uf}(\mathbb{B})$
(b) $(-a_i) \times a_i$ is an open subset of $\mathrm{uf}(\mathbb{B}) \times \mathrm{uf}(\mathbb{B})$.

Lastly,

(c) if $i < j < \lambda$ <u>then</u> $p_j \notin (-a_i) \times a_i$ because $(-a_i) \notin p_j^0$ or $a_i \notin p_j^1$ as $a_i \in \mathbb{B}_j$ and $p_j^0 \cap B_j = p_j^1 \cap \mathbb{B}_j$ by the choice of p_j^0, p_0^1

(d) if $i < j < \lambda$ then $p_i \notin (-a_j) \times a_j$ because $p_i^1 \notin a_j$ by the choice of a_j.

So $\langle (p_i, (-a_i) \times a_i) : i < \lambda) \rangle$ exemplify $\mathrm{dis}(\mathrm{uf}(\mathbb{B}) \times \ \mathrm{uf}(\mathbb{B})) \geq \lambda$ as required.